Section 17.1

Green's Theorem

Line Integral Summary

1 Sketch of the Proof

Closed Curves and Contour Integrals

Let $\mathcal{C}$ be a curve parametrized by $\vec{r}$ on domain $[a, b]$.

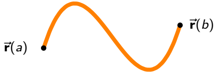
$\mathcal{C}$ is **closed** if $\vec{r}$ begins and ends at the same point: $\vec{r}(a) = \vec{r}(b)$.

$\mathcal{C}$ is **simple** if it does not intersect itself: $\vec{r}$ is one-to-one otherwise.

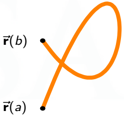
A line integral around a closed curve $\mathcal{C}$ is called a **contour integral**, and denoted by a special symbol:

$$\oint_{\mathcal{C}} f\, ds \quad \text{or} \quad \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

Note: $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ if $\vec{F}$ is conservative.



Not Closed but Simple



Not Closed, Not Simple

Closed but Not Simple

Closed and Simple

Orienting Closed Curves

Every simple closed curve $\mathcal{C}$ in $\mathbb{R}^2$ is the boundary of some region $\mathcal{D}$. We write $\mathcal{C} = \partial\mathcal{D}$. (Here the symbol ∂ means “boundary”.)

There are two ways to orient a simple closed curve $\mathcal{C} = \partial\mathcal{D}$, called **counterclockwise** and **clockwise**, depending on whether $\mathcal{D}$ is on your **left** or your **right** as you walk around $\mathcal{C}$.

Typically we use the **counterclockwise orientation** (as in the standard parametrization of the unit circle).

Example 1, Contour Integrals, $\int_{\mathcal{C}} Pdx + Qdy$ Notation

Let $\mathcal{C}$ be the circle of radius R with standard parametrization $\underbrace{x = R\cos(t),\ y = R\sin(t)}_{\vec{r}(t)}$. Let $\vec{F}(x, y) = \langle y, x \rangle$

Then $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \oint_{\mathcal{C}} Pdx + Qdy$ is computed:

$$\oint_{\mathcal{C}} x\, dy = \int_0^{2\pi} R\cos(t)(R\cos(t))\, dt = \left.\frac{R^2(\cos(t)\sin(t) + t)}{2}\right|_0^{2\pi} = \pi R^2$$

$$\oint_{\mathcal{C}} y\, dx = \int_0^{2\pi} R\sin(t)(-R\sin(t))\, dt = \left.\frac{R^2(\cos(t)\sin(t) - t)}{2}\right|_0^{2\pi} = -\pi R^2$$

Now

$$\oint_{\mathcal{C}} x\, dy + \oint_{\mathcal{C}} y\, dx = \oint_{\mathcal{C}} x\, dy + y\, dx = 0$$

Note: if $f(x, y) = xy$, then $\nabla f(x, y) = \langle y, x \rangle$, we can also find this using Fundamental Theorem of Conservative Field.

Green's Theorem

If $\mathcal{D}$ is a domain in $\mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is a simple, closed curve with counterclockwise orientation, then

$$\oint_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

If $\vec{\mathsf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle$, then $\mathrm{curl}(\vec{\mathsf{F}}) = \langle 0, 0, Q_x - P_y \rangle$.
Writing $\mathrm{curl}_z(\vec{\mathsf{F}})$ for $Q_x - P_y$, we can restate Green's Theorem as follows.

Green's Theorem (Alternative Notation)

If $\mathcal{D}$ is a domain in $\mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is a simple, closed curve with counterclockwise orientation, then

$$\oint_{\partial\mathcal{D}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \iint_{\mathcal{D}} \underbrace{\mathrm{curl}_z(\vec{\mathsf{F}})}_{\mathrm{curl}(\vec{\mathsf{F}})\cdot\vec{\mathsf{k}}}\, dA = \iint_{\mathcal{D}} (\nabla \times \vec{\mathsf{F}}) \cdot \vec{\mathsf{k}}\, dA.$$

Green's Theorem

If $\partial\mathcal{D}$ is a simple, closed curve with counterclockwise orientation, then

$$\oint_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

Example 2: Let $\mathcal{D}$ be the unit disk and $\mathcal{C} = \partial\mathcal{D}$ be the unit circle, with counterclockwise orientation. **Verify** Green's Theorem for $\vec{F}(x,y) = \langle xy^2, x\rangle$.

<u>Solution:</u>

Evaluating the line integral using the parametrization

$$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle,\ \oint_{\partial\mathcal{D}} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \overbrace{-\cos(t)\sin^3(t)}^{u=\sin(t),\ du=\cos(t)dt} + \overbrace{\cos^2(t)}^{=\frac{1+\cos(2t)}{2}}\, dt$$

$$= \int_1^1 -u^3du + \int_0^{2\pi} \frac{1}{2}dt + \int_0^{2\pi} \frac{cos(2t)dt}{2} = \pi$$

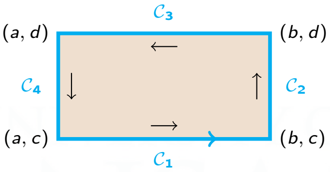
Meanwhile, the right-hand-side of Green's Theorem gives

$$\iint_{\mathcal{D}} \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}\left(xy^2\right)\ dA = \underbrace{\iint_{\mathcal{D}} 1 - 2xy\ dA}_{\int_0^{2\pi}\int_0^1 (1-2r^2\cos(\theta)\sin(\theta))r\,dr\,d\theta}$$

$$= \int_0^{2\pi}\int_0^1 \left(r - r^3\sin(2\theta)\right) dr\,d\theta = \pi$$

Verifying Green's Theorem on Rectangles (Optional)

Let $\mathcal{D} = [a, b] \times [c, d]$, verify the Green's Theorem for vector fields $\vec{F} = \langle P, Q \rangle$ over region $\mathcal{D}$ where $\mathcal{C}$ is the boundary of $\mathcal{D}$, traversed counterclockwise.



$\mathcal{C}$ consists of four smooth pieces: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_a^b \frac{\partial Q}{\partial x}\, dx\, dy = \int_c^d Q(b, y) - Q(a, y)\, dy$$

$$= \int_c^d Q(b, y)\, dy + \int_d^c Q(a, y)\, dy = \int_{\mathcal{C}_2} Q(x, y)\, dy + \int_{\mathcal{C}_4} Q(x, y)\, dy$$

$$= \underbrace{\int_{\mathcal{C}_2} P(x, y)\, dx}_{\text{is zero (since } x(t) = b)} + \int_{\mathcal{C}_2} Q(x, y)\, dy + \underbrace{\int_{\mathcal{C}_4} P(x, y)\, dx}_{\text{is zero (since } x(t) = a)} + \int_{\mathcal{C}_4} Q(x, y)\, dy$$

$$= \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} + \int_{\mathcal{C}_4} \vec{F} \cdot d\vec{r}$$

Verifying Green's Theorem on Rectangles - Cont.

Similarly,

$$-\iint_{\mathcal{D}} \frac{\partial P}{\partial y}\, dA = \int_{\mathcal{C}_1} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} + \int_{\mathcal{C}_3} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}}$$

Comparing the two results:

$$\iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\, dA - \iint_{\mathcal{D}} \frac{\partial P}{\partial y}\, dA = \int_{\mathcal{C}_1} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} + \int_{\mathcal{C}_2} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} + \int_{\mathcal{C}_3} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} + \int_{\mathcal{C}_4} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}}$$

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\mathcal{C}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}}$$

✓

Proof of Green's Theorem (Optional)

Green's Theorem

If $\partial\mathcal{D}$ is a simple, closed curve with counterclockwise orientation, then

$$\oint_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

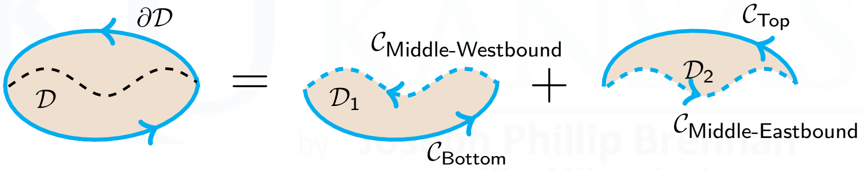
Here is an overview:

1. Case 1 (Rectangular Regions): We have already verified the theorem in the case when $\mathcal{D}$ is a **rectangle**.
2. Case 2 (Rectilinear Regions): Next we discuss the case when $\mathcal{D}$ can be partitioned into rectangles . This case is a direct consequence of additive property on the next slide.
3. Case 3 (General Regions): Last, we claim that any general region can be approximated by rectilinear regions.

Additive Circulation Property

If a domain $\mathcal{D}$ is decomposed into non-overlapping domains $\mathcal{D}_1$ and $\mathcal{D}_2$ that intersect only on part of their boundaries, then

$$\oint_{\partial\mathcal{D}} \vec{F}\cdot d\vec{r} = \oint_{\partial\mathcal{D}_1} \vec{F}\cdot d\vec{r} + \oint_{\partial\mathcal{D}_2} \vec{F}\cdot d\vec{r}$$



$$\oint_{\partial\mathcal{D}_1} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_{Bottom}} \vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_{Middle-Westbound}} \vec{F}\cdot d\vec{r}$$

$$\oint_{\partial\mathcal{D}_2} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_{Top}} \vec{F}\cdot d\vec{r} + \underbrace{\int_{\mathcal{C}_{Middle-Eastbound}} \vec{F}\cdot d\vec{r}}_{-\int_{\mathcal{C}_{Middle-Westbound}} \vec{F}\cdot d\vec{r}}$$

Proof of Green's Theorem (Optional)

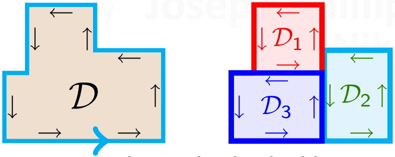
Case 2 (rectilinear regions):

If $\mathcal{D}$ can be partitioned into rectangles $\mathcal{D}_1, \ldots, \mathcal{D}_n$, then we know that

$$\iint_{\mathcal{D}} \nabla \times \vec{\mathsf{F}} \cdot \vec{\mathsf{k}}\, dA \;=\; \iint_{\mathcal{D}_1} \nabla \times \vec{\mathsf{F}} \cdot \vec{\mathsf{k}}\, dA \;+\; \cdots \;+\; \iint_{\mathcal{D}_n} \nabla \times \vec{\mathsf{F}} \cdot \vec{\mathsf{k}}\, dA.$$

In fact, if we orient all boundaries counterclockwise, then

$$\oint_{\partial\mathcal{D}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} \;=\; \oint_{\partial\mathcal{D}_1} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} \;+\; \cdots \;+\; \oint_{\partial\mathcal{D}_n} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}}.$$

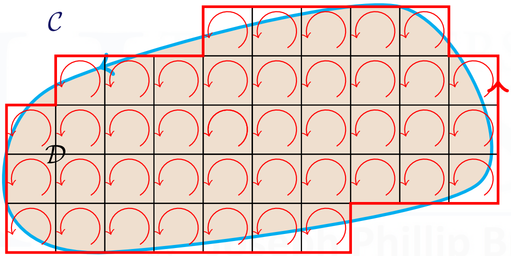


Jeremy Martin has contributed to this slide.

Note that this is possible because the line integral on overlapping pieces of boundaries are additive inverse of each other.

Proof of Green's Theorem (Optional)

Case 3 (General regions):



By approximating the region by smaller and smaller rectangles we conclude that the double integrals of curl on the rectangles adds up to the line integral of the boundary.

2 Applications

Application Examples, Finding the Work

Green's Theorem

If $\partial\mathcal{D}$ is a simple, closed curve with counterclockwise orientation, then

$$\oint_{\partial\mathcal{D}} \vec{\mathrm{F}} \cdot d\vec{\mathrm{r}} = \iint_{\mathcal{D}} \mathrm{curl}_z(\vec{\mathrm{F}})\, dA$$

Example 3: Let $\vec{\mathrm{F}}(x, y) = \left\langle 3y - e^{\sin(x)},\ 7x + \sqrt{y^4 + 1} \right\rangle$.

Calculate the work done by $\vec{\mathrm{F}}$ on a particle moving once around the unit circle counterclockwise.

<ins>Solution:</ins> $\mathrm{curl}_z(\vec{\mathrm{F}}) = \dfrac{\partial}{\partial x}(7x + \sqrt{y^4 + 1}) - \dfrac{\partial}{\partial y}(3y - e^{\sin(x)}) = 4$.

Therefore, by Green's Theorem,

$$\oint_{\partial\mathcal{D}} \vec{\mathrm{F}} \cdot d\vec{\mathrm{r}} = \iint_{\mathcal{D}} 4\, dA = (4)(\mathrm{area}(\mathcal{D})) = 4\pi.$$

Calculating Areas

Recall that the area of $\mathcal{D}$ equals $\iint_{\mathcal{D}} 1\,dA$.

Using Green's Theorem, the area of $\mathcal{D}$ can be calculated using a vector line integral over a carefully chosen vector field.

Choose $\vec{F} = \langle P, Q \rangle$ so that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$.

(i) $P(x,y) = 0$ and $Q(x,y) = x$ $(\vec{\mathrm{F}}(x,y) = \langle 0,\, x \rangle)$

(ii) $P(x,y) = -y$ and $Q(x,y) = 0$ $(\vec{\mathrm{F}}(x,y) = \langle -y,\, 0 \rangle)$

(iii) $P(x,y) = -\frac{1}{2}y$ and $Q(x,y) = \frac{1}{2}x$ $(\vec{\mathrm{F}}(x,y) = \left\langle -\frac{1}{2}y, \frac{1}{2}x \right\rangle)$

Area Formula via Line Integrals

$$\text{Area}(D) = \oint_{\partial\mathcal{D}} x\,dy = \oint_{\partial\mathcal{D}} -y\,dy = \frac{1}{2}\oint_{\partial\mathcal{D}} x\,dy - y\,dx$$

Application Examples, Area

Example 4: Find the area contained inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

<u>Solution:</u> The ellipse can be parametrized as

$$\vec{r}(t) = \langle a\cos(t),\, b\sin(t)\rangle\,, \quad t \in [0, 2\pi].$$

Note that $\vec{r}(t)$ travels counterclockwise about the ellipse.

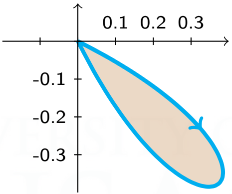
$$\begin{aligned} \text{Area} = \iint_{\mathcal{D}} dA &= \frac{1}{2}\int_{\mathcal{C}} \langle -y, x\rangle \cdot d\vec{r} \\ &= \frac{1}{2}\int_0^{2\pi} -(-b\sin(t))(a\sin(t)) + (a\cos(t))(b\cos(t))dt \\ &= \frac{ab}{2}\int_0^{2\pi} 1\, dt = \pi ab \end{aligned}$$

Application Examples, Area

Example 5: Find the area contained inside the simple closed curve defined by

$$\vec{r}(t) = \langle t(t-1)(t-2),\ t(t-1)(t+1)\rangle$$

for $0 \le t \le 1$.



<u>Solution:</u> The region inside is doubly simple, but it is difficult to impossible to express it in a form suitable for an iterated integral. Instead, use Green's theorem:

$$\begin{aligned}\text{Area} &= \left|\oint_{\mathcal{C}} x\,dy\right| = \left|\int_0^1 x\,\frac{dy}{dt}\,dt\right| \\ &= \left|\int_0^1 (t^3 - 3t^2 + 2t)(3t^2 - 1)\,dt\right| \\ &= \left|\int_0^1 3t^5 - 9t^4 + 5t^3 + 3t^2 - 2t\,dt\right| = \left|-\frac{1}{20}\right|.\end{aligned}$$

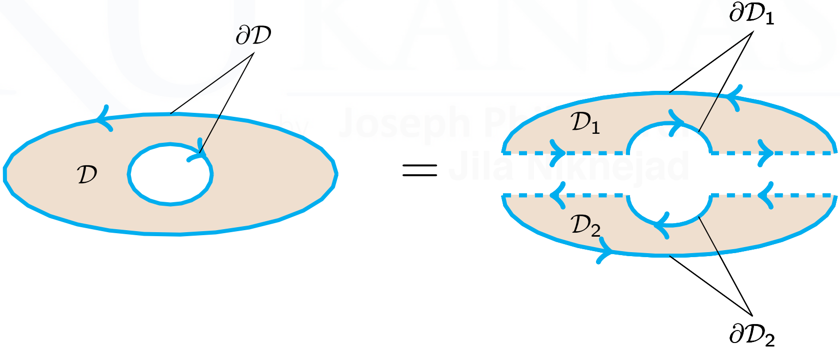
The area is 1/20. (The minus sign indicates that the original parametrization must have traversed the region clockwise.)

3 Regions with Holes

Regions with Holes

For a connected region with holes, the boundary consists of two or more closed curves. Every part of the boundary must be oriented to keep the region on the **left**.

- Outside boundary: counterclockwise.
- Inside boundary: clockwise.

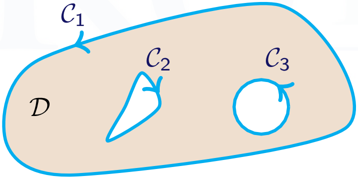


Video

Green's Theorem for Regions with Possible Holes

If $\mathcal{D}$ is a domain in $\mathbb{R}^2$ whose boundary consists of two or more closed curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, each oriented so that $\mathcal{D}$ is on its left, then

$$\oint_{\mathcal{C}_1} P\,dx + Q\,dy + \cdots + \oint_{\mathcal{C}_n} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$



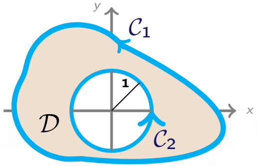
If $\mathcal{D}$ is the region shown at left and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are the curves oriented as shown, then

$$\underbrace{\partial\mathcal{D} = \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_3.}_{\text{This is a mnemonic.}}$$

In this case $\iint_{\mathcal{D}} (Q_x - P_y)\,dA = \left[\oint_{\mathcal{C}_1} + \oint_{\mathcal{C}_2} - \oint_{\mathcal{C}_3} \right] P\,dx + Q\,dy.$

Domain With Holes Example

Example 6: Suppose that the region $\mathcal{D}$ shown at right has area 8. Calculate $\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \langle x - y,\ x + y^3 \rangle$.



Solution: By Green's Theorem,

$$\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} - \oint_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{D}} \text{curl}_z(\vec{F})\, dA = \iint_{\mathcal{D}} 2\, dA = 16.$$

Parametrizing $\mathcal{C}_2$ as $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$,

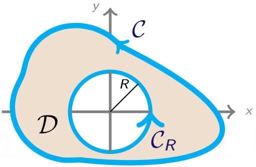
$$\oint_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1 \underbrace{- \sin(t)\cos(t) + \sin^3(t)\cos(t)}_{\text{Use u-sub } u=\sin(t) \text{ or by symmetry}}\, dt = 2\pi.$$

Therefore, $\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} = \iint_D \text{curl}_z(\vec{F})\, dA + \oint_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} = 16 + 2\pi.$ ▸ Video

Winding Numbers (Optional)

Example 7: Calculate $\oint_{\mathcal{C}} \vec{\mathsf{F}}_{\text{vor}} \cdot d\vec{\mathsf{r}}$, where

$$\vec{\mathsf{F}}_{\text{vor}}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$



<u>Solution:</u> By Green's Theorem,

$$\oint_{\mathcal{C}} \vec{\mathsf{F}}_{\text{vor}} \cdot d\vec{\mathsf{r}} - \oint_{\mathcal{C}_R} \vec{\mathsf{F}}_{\text{vor}} \cdot d\vec{\mathsf{r}} = \iint_{\mathcal{D}} \text{curl}_z(\vec{\mathsf{F}}_{\text{vor}})\, dA$$

Since $\text{curl}_z(\vec{\mathsf{F}}_{\text{vor}}) = 0$, we conclude that

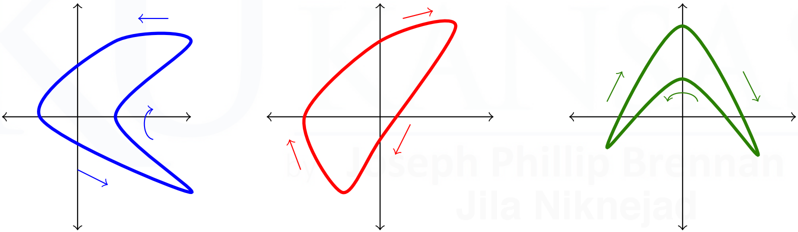
$$\oint_{\mathcal{C}} \vec{\mathsf{F}}_{\text{vor}}\, d\vec{\mathsf{r}} = \oint_{\mathcal{C}_R} \vec{\mathsf{F}}_{\text{vor}} \cdot d\vec{\mathsf{r}} = 2\pi.$$

If $\mathcal{C}$ is a closed curve not passing through the origin, its **winding number** is defined as $\dfrac{1}{2\pi} \oint_{\mathcal{C}} \vec{\mathsf{F}}_{\text{vor}} \cdot d\vec{\mathsf{r}}$.

Winding Numbers (Optional)

If $\mathcal{C}$ is a closed curve not passing through the origin, its **winding number** (which is always an integer!) is defined as

$$w(\mathcal{C}) = \frac{1}{2\pi} \oint_{\mathcal{C}} \vec{\mathsf{F}}_{\mathsf{vor}} \cdot d\vec{\mathsf{r}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}.$$



$w(\mathcal{C}) = 1$ $w(\mathcal{C}) = -1$ $w(\mathcal{C}) = 0$

- If a clock is placed at the origin, and the minute hand moves so that it always points to $\vec{\mathsf{r}}(t)$, then the number of clockwise revolutions of the hour hand during one trip around $\mathcal{C}$ is $-w(\mathcal{C})$.

The widning number content was contributed by Prof. Martin.